微積分作業：Hw1

September 12, 2025

題目

1. $\lim\limits_{x\to 3} \dfrac{\sqrt{x+1}-2}{x-3}$

2. $\lim\limits_{x\to 0} \dfrac{\sin(5x)}{2x}$

3. $\lim\limits_{x\to 0} x\cos\left(\dfrac{1}{x}\right)$　(提示：夾擠定理)

解答

1. $\lim\limits_{x\to 3} \dfrac{\sqrt{x+1}-2}{x-3}$

分子分母同乘 $\sqrt{x+1}+2$：

$$\frac{\sqrt{x+1}-2}{x-3}\cdot\frac{\sqrt{x+1}+2}{\sqrt{x+1}+2} = \frac{(x+1)-4}{(x-3)(\sqrt{x+1}+2)} = \frac{x-3}{(x-3)(\sqrt{x+1}+2)} = \frac{1}{\sqrt{x+1}+2}.$$

令 $x\to 3$，得

$$\lim_{x\to 3}\frac{\sqrt{x+1}-2}{x-3} = \frac{1}{\sqrt{3+1}+2} = \frac{1}{2+2} = \boxed{\tfrac{1}{4}}.$$

2. $\lim\limits_{x\to 0} \dfrac{\sin(5x)}{2x}$

寫成標準極限的比例：

$$\frac{\sin(5x)}{2x} = \frac{\sin(5x)}{5x}\cdot\frac{5}{2} \quad\Longrightarrow\quad \lim_{x\to 0}\frac{\sin(5x)}{2x} = \left(\lim_{x\to 0}\frac{\sin(5x)}{5x}\right)\cdot\frac{5}{2} = 1\cdot\frac{5}{2} = \boxed{\tfrac{5}{2}}.$$

3. $\lim\limits_{x\to 0} x \cos\left(\dfrac{1}{x}\right)$

因為對所有 $x \neq 0$ 皆有 $-1 \leq \cos(1/x) \leq 1$，故

$$-|x| \leq x \cos\left(\frac{1}{x}\right) \leq |x|.$$

又 $\lim\limits_{x\to 0}(-|x|) = 0 = \lim\limits_{x\to 0}|x|$，由夾擠定理可得

$$\lim\limits_{x\to 0} x \cos\left(\frac{1}{x}\right) = \boxed{0}.$$